

$3nj$ -SYMBOLS AND IDENTITIES FOR q -BESSEL FUNCTIONS

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ABSTRACT. The $6j$ -symbols for representations of the q -deformed algebra of polynomials on $SU(2)$ are given by Jackson's third q -Bessel functions. This interpretation leads to several summation identities for the q -Bessel functions. Multivariate q -Bessel functions are defined, which are shown to be limit cases of multivariate Askey-Wilson polynomials. The multivariate q -Bessel functions occur as $3nj$ -symbols.

1. INTRODUCTION

It is well known that Wigner's $6j$ -symbols for the $SU(2)$ group are multiples of hypergeometric orthogonal polynomials called the Racah polynomials. Similarly, $6j$ -symbols for the $SU(2)$ quantum group can be expressed in terms of q -Racah polynomials, which are q -hypergeometric orthogonal polynomials. With this interpretation, properties of $6j$ -symbols such as summation formulas and orthogonality relations, lead to properties of specific families of orthogonal polynomials.

In this paper we consider $6j$ -symbols for representations of the q -deformed algebra of polynomials on $SU(2)$. This algebra has as irreducible representations the trivial one, and a family of infinite dimensional representations which disappear in the classical limit. The $6j$ -symbols for tensor products of the three infinite dimensional representations can be expressed in terms of Jackson's third q -Bessel functions [7]. Note that, different from the classical $6j$ -symbols, these are not polynomials. We consider three fundamental identities for $6j$ -symbols (see e.g. [1]): Racah's backcoupling identity, the Biedenharn-Elliott identity and the hexagon identity. These identities are obtained by decomposing 3- or 4-fold tensor product representations in several ways. To keep track of the order of decomposing the representations, it is convenient to identify certain vectors in the representations spaces with binary trees. Then the $6j$ -symbols can be considered as coupling coefficients between two of these trees. The identities we obtain can be interpreted as summation identities for q -Bessel functions. We remark that the hexagon identity implies that the q -Bessel functions are matrix elements of an infinite dimensional solution of the quantum Yang-Baxter equation (or, the star-triangle equation in IRF-models), see e.g. [10], which should be of independent interest.

We also consider specific $3nj$ -symbols, which may naturally be considered as multivariate q -Bessel functions. The one variable q -Bessel functions fit into an extended Askey-scheme [13] of orthogonal q -hypergeometric functions; the original (q -)Askey-scheme [12] consists of (q -)hypergeometric orthogonal polynomials. We will show that the multivariate q -Bessel

functions fit into an extended Askey-scheme of multivariate orthogonal functions of q -hypergeometric type, by showing that the multivariate q -Bessel functions can be obtained as limits of the multivariate Askey-Wilson polynomials defined by Gasper and Rahman [4], which are the q -analogs of Tratnik's multivariate Wilson polynomials [19]. The multivariate Askey-Wilson polynomials can be thought of as being on top of a scheme of multivariate orthogonal polynomials; several limit cases are considered in [4], [5], [8]. Geronimo and Illiev [6] obtained multivariate Askey-Wilson functions generalizing the multivariate Askey-Wilson functions, which should be on top of the extended Askey-scheme. Several families of orthogonal polynomials in this scheme and its $q = 1$ analog are connected to tensor product representations and binary coupling schemes, see e.g. Van der Jeugt [9], Rosengren [17] and Scarabotti [18].

This paper is organized as follows. In Section 2 the quantum algebra $\mathcal{A}_q(\text{SU}(2))$ and its representation theory is recalled. In section 3 it is shown that the $6j$ -symbols are essentially q -Bessel functions, using a generating function for q -Bessel functions. Using binary trees we obtain the fundamental identities for $6j$ -symbols, leading to summation formulas for the q -Bessel functions. In Section 4 we first define multivariate q -Bessel functions as nontrivial products of q -Bessel functions, and we prove orthogonality relations. Then we show that these multivariate q -Bessel functions occur as $3nj$ -symbols, and use this interpretation to find a summation formula.

Notations. We use $\mathbb{N} = \{0, 1, 2, \dots\}$ and we use standard notation for q -hypergeometric functions as in [3].

2. THE QUANTUM ALGEBRA $\mathcal{A}_q(\text{SU}(2))$

Let $q \in (0, 1)$. The q -deformed algebra of polynomials on $\text{SU}(2)$ is the complex unital associative algebra $\mathcal{A}_q = \mathcal{A}_q(\text{SU}(2))$ generated by $\alpha, \beta, \gamma, \delta$, which satisfy the relations

$$(2.1) \quad \begin{aligned} \alpha\beta &= q\beta\alpha, & \alpha\gamma &= q\gamma\alpha, & \beta\delta &= q\delta\beta, & \gamma\delta &= q\delta\gamma, \\ \beta\gamma &= \gamma\beta, & \alpha\delta - q\beta\gamma &= 1 = \delta\alpha - q^{-1}\beta\gamma. \end{aligned}$$

\mathcal{A}_q is a Hopf- $*$ -algebra with $*$ -structure and comultiplication Δ defined on the generators by

$$(2.2) \quad \alpha^* = \delta, \quad \beta^* = -q\gamma, \quad \gamma^* = -q^{-1}\beta, \quad \delta^* = \alpha,$$

$$(2.3) \quad \begin{aligned} \Delta(\alpha) &= \alpha \otimes \alpha + \beta \otimes \gamma, & \Delta(\beta) &= \alpha \otimes \beta + \beta \otimes \delta, \\ \Delta(\gamma) &= \gamma \otimes \alpha + \delta \otimes \gamma, & \Delta(\delta) &= \delta \otimes \delta + \gamma \otimes \beta. \end{aligned}$$

The irreducible $*$ -representation of \mathcal{A}_q are either 1-dimensional, or infinite dimensional. The infinite dimensional irreducible $*$ -representations are labeled by $\phi \in [0, 2\pi)$, and we denote a representation by π_ϕ . The representation space of π_ϕ is $\ell^2(\mathbb{N})$. The generators

$\alpha, \beta, \gamma, \delta$ act on the standard orthonormal basis $\{e_n \mid n \in \mathbb{N}\}$ of $\ell^2(\mathbb{N})$ by

$$\begin{aligned}\pi_\phi(\alpha) e_n &= \sqrt{1 - q^{2n}} e_{n-1}, \\ \pi_\phi(\beta) e_n &= -e^{-i\phi} q^{n+1} e_n, \\ \pi_\phi(\gamma) e_n &= e^{i\phi} q^n e_n, \\ \pi_\phi(\delta) e_n &= \sqrt{1 - q^{2n+2}} e_{n+1}.\end{aligned}$$

Note that $\pi_\phi(\gamma\beta)$ is a self-adjoint diagonal operator in the standard basis.

Remark 2.1. In this paper we consider tensor products of π_0 . We could also consider the representation $\pi_{\phi_1} \otimes \pi_{\phi_2}$, but this would not lead to more general results in this paper, because representation labels only occur in phase factors, see [7, §II.A].

Let $\sigma : \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$ be the flip operator, the linear operator defined on pure tensors by $\sigma(v_1 \otimes v_2) = v_2 \otimes v_1$. We write

$$\pi_{12} = (\pi_0 \otimes \pi_0) \Delta, \quad \pi_{21} = \sigma \pi_{12} \sigma.$$

For three-fold tensor product representations we write

$$\pi_{1(23)} = (\pi_0 \otimes \pi_0 \otimes \pi_0)(1 \otimes \Delta)(\Delta), \quad \pi_{(12)3} = (\pi_0 \otimes \pi_0 \otimes \pi_0)(\Delta \otimes 1)(\Delta).$$

Since Δ is coassociative, we have $\pi_{1(23)} = \pi_{(12)3}$.

From (2.3) one finds

$$\Delta(\gamma\gamma^*) = q^{-1} \Delta(\gamma\beta) = -q^{-1} (\gamma\beta \otimes \alpha\delta + \gamma\alpha \otimes \alpha\beta + \delta\beta \otimes \gamma\delta + \delta\alpha \otimes \gamma\beta).$$

Using this, eigenvectors of $\pi_{12}(\gamma\gamma^*)$ can be computed (see [7] for details): for $p \in \mathbb{Z}$ and $x \in \mathbb{N}$ define

$$e_{x,p}^{12} = \sum_{\substack{n,m \in \mathbb{N} \\ n-m=p}} C_{x,m,n} e_m \otimes e_n,$$

where we assume $e_{-n} = 0$ for $n \geq 1$, then $\pi_{12}(\gamma\gamma^*) e_{x,p}^{12} = q^{2x} e_{x,p}^{12}$. The Clebsch-Gordan coefficients $C_{x,m,n}$ can be given explicitly in terms of Wall polynomials, see [12], which are defined by

$$\begin{aligned}(2.4) \quad p_n(q^x; a; q) &= {}_2\varphi_1 \left(\begin{matrix} q^{-n}, 0 \\ aq \end{matrix}; q, q^{x+1} \right) \\ &= \frac{(-a)^n q^{\frac{1}{2}n(n+1)}}{(aq; q)_n} {}_2\varphi_0 \left(\begin{matrix} q^{-n}, q^{-x} \\ - \end{matrix}; q, \frac{q^x}{a} \right),\end{aligned}$$

for $n, x \in \mathbb{N}$. The second expression follows from applying transformation [3, (III.8)] with $b \rightarrow 0$. Note that, for $x \in \mathbb{N}$, the ${}_2\varphi_0$ -series can be considered as a polynomial in q^{-n} of degree x . This polynomial is (proportional to) an Al-Salam–Carlitz II polynomial.

Let the function $\bar{p}_n(q^x; a; q)$ be defined by

$$(2.5) \quad \bar{p}_n(q^x; a; q) = (-1)^{n+x} \sqrt{\frac{(aq)^{x-n} (aq; q)_\infty (aq; q)_n}{(q; q)_n (q; q)_x}} p_n(q^x; a; q),$$

then from the orthogonality relation for the Wall polynomials and from completeness we obtain the orthogonality relations

$$\sum_{x \in \mathbb{N}} \bar{p}_n(q^x; a; q) \bar{p}_m(q^x; a; q) = \delta_{nm}, \quad \sum_{n \in \mathbb{N}} \bar{p}_n(q^x; a; q) \bar{p}_n(q^y; a; q) = \delta_{xy},$$

for $0 < a < q^{-1}$. The second relation corresponds to orthogonality relations for Al-Salam–Carlitz II polynomials. The coefficients $C_{x,m,n}$, $m, n \in \mathbb{N}$, are defined by

$$C_{x,m,n} = \begin{cases} \bar{p}_n(q^{2x}; q^{2(n-m)}; q^2), & n \geq m, \\ \bar{p}_m(q^{2x}; q^{2(m-n)}; q^2), & n \leq m, \end{cases}$$

and they satisfy

$$(2.6) \quad C_{x,n,m} = C_{x,m,n},$$

which follows from the explicit expression as a ${}_2\varphi_1$ -function. Furthermore, we define $C_{x,m,n} = 0$ for $m \in -\mathbb{N}_{\geq 1}$ or $n \in -\mathbb{N}_{\geq 1}$ or $x \in -\mathbb{N}_{\geq 1}$.

The set $\{e_{x,p}^{12} \mid p \in \mathbb{Z}, x \in \mathbb{N}\}$ is an orthonormal basis for $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$. The actions of the \mathcal{A}_q -generators on this basis are given by

$$(2.7) \quad \begin{aligned} \pi_{12}(\alpha) e_{x,p}^{12} &= \sqrt{1 - q^{2x}} e_{x-1,p}^{12}, \\ \pi_{12}(\beta) e_{x,p}^{12} &= -q^{x+1} e_{x,p+1}^{12}, \\ \pi_{12}(\gamma) e_{x,p}^{12} &= q^x e_{x,p-1}^{12}, \\ \pi_{12}(\delta) e_{x,p}^{12} &= \sqrt{1 - q^{2x+2}} e_{x+1,p}^{12}, \end{aligned}$$

where $e_{-1,p}^{12} = 0$. We can also find eigenvectors $e_{x,p}^{21}$ of $\pi_{21}(\gamma\gamma^*)$ for eigenvalue q^{2x} , $x \in \mathbb{N}$:

$$e_{x,p}^{21} = \sum_{\substack{n,m \in \mathbb{N} \\ m-n=p}} C_{x,m,n} e_m \otimes e_n = e_{x,-p}^{12}.$$

3. $6j$ -SYMBOLS AND q -BESSEL FUNCTIONS

In [7] explicit expressions for the $6j$ -symbols (and for more general coupling coefficients) have been found. It turns out that they are essentially q -Bessel functions. Here we derive these results again using a more direct approach, and use this interpretation of the q -Bessel functions to obtain summation identities.

3.1. $6j$ -symbols. In the same way as above we can find eigenvectors of $\pi_{1(23)}(\gamma\gamma^*)$ and $\pi_{(12)3}(\gamma\gamma^*)$; for $x \in \mathbb{N}$, $p, r \in \mathbb{Z}$,

$$\begin{aligned} e_{x,p,r}^{1(23)} &= \sum_{n \in \mathbb{N}} C_{x,n,n+p} e_n \otimes e_{n+p,x-n-r}^{23} \\ &= \sum_{n,m \in \mathbb{N}} C_{x,n,n+p} C_{n+p,m,k} e_n \otimes e_m \otimes e_k, & n - m + k = x - r, \\ e_{x,p,r}^{(12)3} &= \sum_{k \in \mathbb{N}} C_{x,k-p,k} e_{k-p,r-x+k}^{12} \otimes e_k \\ &= \sum_{k,m \in \mathbb{N}} C_{x,k-p,k} C_{k-p,n,m} e_n \otimes e_m \otimes e_k, & n - m + k = x - r, \end{aligned}$$

are eigenvectors for eigenvalue q^{2x} , $x \in \mathbb{N}$. We use here the convention $e_{-n} = e_{-n,p} = 0$ for $n \in -\mathbb{N}_{\geq 1}$. The actions of the \mathcal{A}_q -generators $\alpha, \beta, \gamma, \delta$ on the eigenvectors can be obtained in the same way as in [7]

$$\begin{aligned} \pi_{1(23)}(\alpha) e_{x,p,r}^{1(23)} &= \sqrt{1 - q^{2x}} e_{x-1,p,r}^{1(23)}, & \pi_{(12)3}(\alpha) e_{x,p,r}^{(12)3} &= \sqrt{1 - q^{2x}} e_{x-1,p,r}^{(12)3}, \\ \pi_{1(23)}(\beta) e_{x,p,r}^{1(23)} &= -q^{x+1} e_{x,p+1,r}^{1(23)}, & \pi_{(12)3}(\beta) e_{x,p,r}^{(12)3} &= -q^{x+1} e_{x,p+1,r}^{(12)3}, \\ \pi_{1(23)}(\gamma) e_{x,p,r}^{1(23)} &= q^x e_{x,p-1,r}^{1(23)}, & \pi_{(12)3}(\gamma) e_{x,p,r}^{(12)3} &= q^x e_{x,p-1,r}^{(12)3}, \\ \pi_{1(23)}(\delta) e_{x,p,r}^{1(23)} &= \sqrt{1 - q^{2x+2}} e_{x+1,p,r}^{1(23)}, & \pi_{(12)3}(\delta) e_{x,p,r}^{(12)3} &= \sqrt{1 - q^{2x+2}} e_{x+1,p,r}^{(12)3}, \end{aligned}$$

where $e_{-1,p,r} = 0$. Note that this corresponds exactly to the actions on the eigenvectors $e_{x,p}$.

The $6j$ -symbol (or Racah coefficient) $R_{p_1,r_1;p_2,r_2}^x$ is the (re)coupling coefficient between the two eigenvectors;

$$R_{p_1,r_1;p_2,r_2}^x = \langle e_{x,p_1,r_1}^{1(23)}, e_{x,p_2,r_2}^{(12)3} \rangle,$$

or equivalently

$$(3.1) \quad e_{x,p_1,r_1}^{1(23)} = \sum_{p_2,r_2} R_{p_1,r_1;p_2,r_2}^x e_{x,p_2,r_2}^{(12)3}.$$

We start by looking at some simple properties of R .

Proposition 3.1. *The coefficients R have the following properties:*

- (i) *Orthogonality relations:* $\sum_{p_1,r_1 \in \mathbb{Z}} R_{p_1,r_1;p_2,r_2}^x R_{p_1,r_1;p_3,r_3}^x = \delta_{p_2,p_3} \delta_{r_2,r_3}.$
- (ii) $R_{p_1,r_1;p_2,r_2}^x = R_{p_1+k,r_1;p_2+k,r_2}^x$ for $k \in \mathbb{Z}$.
- (iii) $R_{p_1,r_1;p_2,r_2}^x = R_{p_1,r_1;p_2,r_2}^{x+k}$ for $k \in \mathbb{Z}_{\geq -x}$.
- (iv) For $k, m, n \in \mathbb{N}$,

$$C_{x,n+p_1,n} C_{n+p_1,m,k} = \sum_{p_2 \in \mathbb{Z}_{\leq k}} R_{p_1,r;p_2,r}^x C_{x,k-p_2,k} C_{k-p_2,m,n}, \quad x - r = n - m + k.$$

- (v) *Duality:* $R_{p_1,r;p_2,r}^x = R_{-p_2,r;-p_1,r}^x.$

Note that identity (iii) implies that R is independent of x , therefore we will omit the superscript ‘ x ’.

Proof. The coefficients R are matrix coefficients of a unitary operator, which leads to the orthogonality relations. The next two identities follow from the $*$ -structure of \mathcal{A}_q . From $\beta^* = -q\gamma$ we obtain

$$\langle e_{x,p_1 \pm 1, r_1}^{1(23)}, e_{x,p_2, r_2}^{(12)3} \rangle = \langle e_{x,p_1, r_1}^{1(23)}, e_{x,p_2 \mp 1, r_2}^{(12)3} \rangle,$$

which implies (ii). Identity (iii) follows from $\alpha^* = \delta$. Identity (iv) follow from the expansion

$$e_{x,p_1, r_1}^{1(23)} = \sum_{p_2, r_2 \in \mathbb{Z}} R_{p_1, r_1; p_2, r_2} e_{x,p_2, r_2}^{(12)3}.$$

by taking inner products with $e_n \otimes e_m \otimes e_k$. The duality property follows from identity (iv). \square

3.2. q -Bessel functions. Define

$$J_\nu(x; q) = x^{\frac{\nu}{2}} \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} {}_1\varphi_1 \left(\begin{matrix} 0 \\ q^{\nu+1} \end{matrix}; q, qx \right), \quad x \geq 0, \nu \notin \mathbb{Z}_{\leq -1},$$

which is Jackson’s third q -Bessel function (also known as the Hahn-Exton q -Bessel function), see e.g. [16]. We will need the q -Bessel functions with $\nu \in \mathbb{Z}$; for negative integers they are defined by

$$J_{-n}(x; q) = (-1)^n q^{\frac{n}{2}} J_n(xq^n; q), \quad n \in \mathbb{N},$$

see [16, (2.6)]. We will use the following generating function to identify the $6j$ -symbols with q -Bessel functions.

Proposition 3.2. *For $|t| < 1$,*

$$\sum_{m=0}^{\infty} q^{-\frac{\nu m}{2}} J_\nu(xq^m; q) \frac{t^m}{(q; q)_m} = x^{\frac{\nu}{2}} \frac{(q^{\nu+1}; q)_\infty}{(q, t; q)_\infty} {}_1\varphi_1 \left(\begin{matrix} t \\ q^{\nu+1} \end{matrix}; q, qx \right).$$

Proof. Write J_ν as a ${}_1\varphi_1$ -series, interchange the order of summation, and use summation formula [3, (II.1)];

$$\begin{aligned} \sum_{m=0}^{\infty} {}_1\varphi_1 \left(\begin{matrix} 0 \\ q^{\nu+1} \end{matrix}; q, xq^{m+1} \right) \frac{t^m}{(q; q)_m} &= \sum_{k=0}^{\infty} \frac{(-1)^k q^{\frac{1}{2}k(k-1)} (xq)^k}{(q, q^{\nu+1}; q)_k} \sum_{m=0}^{\infty} \frac{q^{mk} t^m}{(q; q)_m} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k q^{\frac{1}{2}k(k-1)} (xq)^k}{(q, q^{\nu+1}; q)_k (tq^k; q)_\infty}. \end{aligned} \quad \square$$

If $t^{-1}q^{\nu+1} \in q^{-\mathbb{N}}$ the right hand side in the Proposition 3.2 can be written in terms of a Wall polynomial, which gives the following special case.

Corollary 3.3. *For $n \in \mathbb{N}$,*

$$\sum_{m=0}^{\infty} q^{-\frac{1}{2}(\nu-n)m} J_{\nu-n}(xq^m; q) \frac{q^{m(\nu+1)}}{(q; q)_m} = x^{\frac{1}{2}(\nu-n)} \frac{(qx; q)_\infty}{(q; q)_\infty} p_n(q^\nu; x; q).$$

Proof. In Proposition 3.2 replace ν by $\nu - n$, set $t = q^{\nu+1}$, and use the transformation

$${}_1\varphi_1\left(\frac{A}{B}; q, Z\right) = \frac{(A, Z; q)_\infty}{(B; q)_\infty} {}_2\varphi_1\left(\begin{matrix} 0, B/A \\ Z \end{matrix}; q, A\right),$$

(which is a special case of [3, (III.4)]) and the definition (2.4) of the Wall polynomials. \square

We are now in a position to show that the $6j$ -symbols are essentially q -Bessel functions.

Proposition 3.4. For $p_1, p_2, r_1, r_2 \in \mathbb{Z}$,

$$R_{p_1, r_1; p_2, r_2} = \delta_{r_1, r_2} (-q)^{p_1 - p_2} J_{r_1}(q^{2p_1 - 2p_2}; q^2).$$

Proof. We write out Proposition 3.1(iv) for $m = k = 0$, and we replace p_2 by $-p_2$,

$$\sum_{p_2 \in \mathbb{N}} R_{p_1, r; -p_2, r} \frac{(-1)^{p_2} q^{p_2(n+x+1)}}{(q^2; q^2)_{p_2}} = \frac{(-1)^{p_1} q^{p_1(x-n+1)}}{(q^2; q^2)_{p_1}} p_n(q^{2x}; q^{2p_1}; q^2), \quad x - r = n,$$

then the result follows from Corollary 3.3. \square

3.3. Identities. Several classical identities for $6j$ -symbols for $SU(2)$ remain valid for our $6j$ -symbols. By Proposition 3.4 these can be interpreted as identities for q -Bessel functions.

First of all, the orthogonality relations for the $6j$ -symbols from Proposition 3.1 are equivalent to the well-known q -Hankel orthogonality relations, see [16, (2.11)], for the q -Bessel functions J_ν .

Theorem 3.5. For $n, m \in \mathbb{Z}$,

$$\sum_{x \in \mathbb{Z}} J_\nu(q^{x+m}; q) J_\nu(q^{x+n}; q) q^x = \delta_{m, n} q^{-n}.$$

To derive other identities it is convenient to represent eigenvectors of $\gamma\gamma^*$ as binary trees, see e.g. Van der Jeugt's lecture notes [9] for more details. We denote

$$e_{x, n_2 - n_1}^{12} = \begin{array}{c} x \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \\ n_1 \quad n_2 \end{array}$$

where $n_1, n_2, x \in \mathbb{Z}$. Equivalently, we can identify this tree with the Clebsch-Gordan coefficient C_{x, n_1, n_2} , similar as in [18]. The identity $e_{x, p}^{12} = e_{x, -p}^{21}$, which is equivalent to (2.6), is represented as

$$(3.2) \quad \begin{array}{c} x \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \\ n_1 \quad n_2 \end{array} = \begin{array}{c} x \\ \swarrow \quad \searrow \\ \bullet \quad \bullet \\ n_2 \quad n_1 \end{array}$$

where $p = n_1 - n_2$. By coupling two of these we can represent eigenvectors corresponding to three-fold tensor products:

$$e_{x,p_1,r_{123}}^{1(23)} = \begin{array}{c} x \\ \swarrow \quad \searrow \\ n_1 \quad n_2 \quad n_3 \end{array} \quad e_{x,p_2,r_{123}}^{(12)3} = \begin{array}{c} x \\ \swarrow \quad \searrow \\ p'_2 \quad n_1 \quad n_2 \quad n_3 \end{array}$$

where $p'_1 = n_1 + p_1$, $p'_2 = n_3 - p_2$ and $r_{ijk} = x - n_i + n_j - n_k$ for $i, j, k \in \{1, 2, 3\}$. Now we can e.g. represent the identities $e_{x,p_1,r_{123}}^{1(23)} = e_{x,p_1,r_{132}}^{1(32)} = e_{x,-p_1,r_{231}}^{(23)1}$ by

$$\begin{array}{c} x \\ \swarrow \quad \searrow \\ n_1 \quad n_2 \quad n_3 \end{array} = \begin{array}{c} x \\ \swarrow \quad \searrow \\ n_1 \quad n_3 \quad n_2 \end{array} = \begin{array}{c} x \\ \swarrow \quad \searrow \\ p'_1 \quad n_3 \quad n_2 \quad n_1 \end{array}$$

The transition (3.1) from $e_{x,p_1,r}^{1(23)}$ to $e_{x,p_2,r}^{(12)3}$ which involves a $6j$ -symbol, which is equivalent to identity (ii) in Proposition 3.1 in terms of Clebsch-Gordan coefficients, is represented as

$$\begin{array}{c} x \\ \swarrow \quad \searrow \\ n_1 \quad n_2 \quad n_3 \end{array} \xrightarrow{R_{p'_1,p'_2}^{x,n_1,n_2,n_3}} \begin{array}{c} x \\ \swarrow \quad \searrow \\ p'_2 \quad n_1 \quad n_2 \quad n_3 \end{array}$$

where the coefficient R is given by

$$(3.3) \quad R_{p'_1,p'_2}^{x,n_1,n_2,n_3} = R_{p_1,r_{123};p_2,r_{123}} = (-q)^{p'_1+p'_2-n_1-n_3} J_{x-n_1+n_2-n_3}(q^{2p'_1+2p'_2-2n_1-2n_3}; q^2).$$

Note that the transition from right to left involves exactly the same $6j$ -symbol. To find identities for the $6j$ -symbols we can use the binary trees, and identities for these trees as explained above, without refering to the underlying eigenvectors. We obtain the following identities, which can be considered as analogs of Racah's backcoupling identity, the Biedenharn-Elliott (or pentagon) identity and the hexagon identity.

Theorem 3.6. *The following identities hold:*

$$(i) \quad R_{p_1,p_2}^{x,n_1,n_2,n_3} = \sum_{p \in \mathbb{Z}} R_{p_1,p}^{x,n_1,n_3,n_2} R_{p,p_2}^{x,n_3,n_1,n_2},$$

or in terms of q -Bessel functions

$$J_{r_{123}}(q^{p_1+p_2}; q) = \sum_{p \in \mathbb{Z}} J_{r_{132}}(q^{p+p_1}; q) J_{r_{312}}(q^{p+p_2}; q) q^p,$$

where $r_{ijk} = x - n_i + n_j - n_k$.

$$(ii) \quad R_{r_1,p_2}^{x,n_1,n_2,p_1} R_{p_1,r_2}^{x,p_2,n_3,n_4} = \sum_{p \in \mathbb{Z}} R_{p_1,p}^{r_1,n_2,n_3,n_4} R_{r_1,r_2}^{x,n_1,p,n_4} R_{p,p_2}^{r_2,n_1,n_2,n_3},$$

which in terms of q -Bessel functions is equivalent to the product formula

$$J_{\nu+\mu_1}(q^{P-Q}; q) J_{\nu+\mu_2}(q^{Q-R}; q) = \sum_{\mu \in \mathbb{Z}} A_{P,Q,R}^{\mu_1,\mu_2,\mu} J_{\nu+\mu}(q^{P-R}; q)$$

where $P, Q, R, \nu, \mu_1, \mu_2 \in \mathbb{Z}$ and

$$A_{P,Q,R}^{\mu_1,\mu_2,\mu} = (-1)^{\mu_1+\mu_2} q^{\mu-\frac{1}{2}(\mu_1+\mu_2)} J_{\mu_2-\mu_1+P-Q}(q^{\mu-\mu_1}; q) J_{\mu_1-\mu_2+Q-R}(q^{\mu-\mu_2}; q).$$

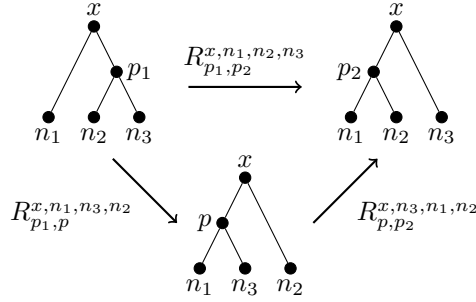
$$(iii) \quad \sum_{r \in \mathbb{Z}} R_{p_2,r}^{x,p_1,n_3,n_4} R_{p_3,p_1}^{r,n_2,n_1,n_3} R_{p_4,r}^{x,p_3,n_2,n_4} = \sum_{r \in \mathbb{Z}} R_{r,p_1}^{x,n_1,n_2,p_2} R_{p_2,p_4}^{r,n_2,n_4,n_3} R_{r,p_3}^{x,n_1,n_3,p_4},$$

or in terms of q -Bessel functions,

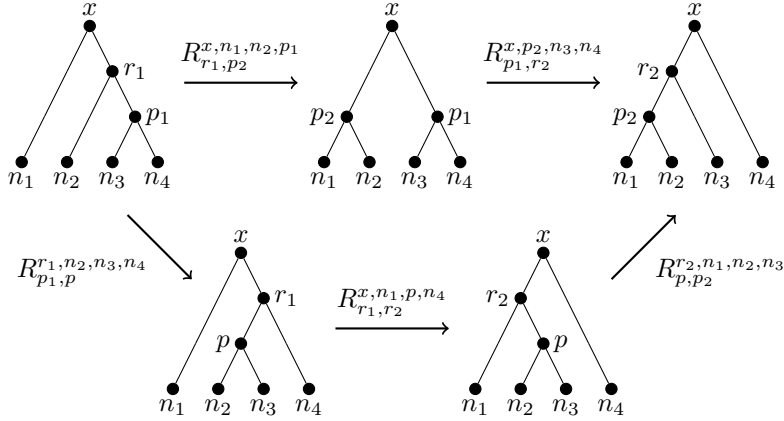
$$\begin{aligned} & \sum_{r \in \mathbb{Z}} (-1)^{p_2+p_4} q^{r-n_4+\frac{1}{2}(p_2+p_4)} J_{r-n_2+n_1-n_3}(q^{p_1+p_3-n_2-n_3}; q) \\ & \quad \times J_{x-p_1+n_3-n_4}(q^{r+p_2-p_1-n_4}; q) J_{x-p_3+n_2-n_4}(q^{r+p_4-p_3-n_4}; q) \\ & = \text{idem}((n_1, n_2, p_1, p_3) \leftrightarrow (n_4, n_3, p_2, p_4)). \end{aligned}$$

Here ‘idem’ means that the same expression is inserted but with the parameters interchanged as indicated.

Proof. The first identity follows from



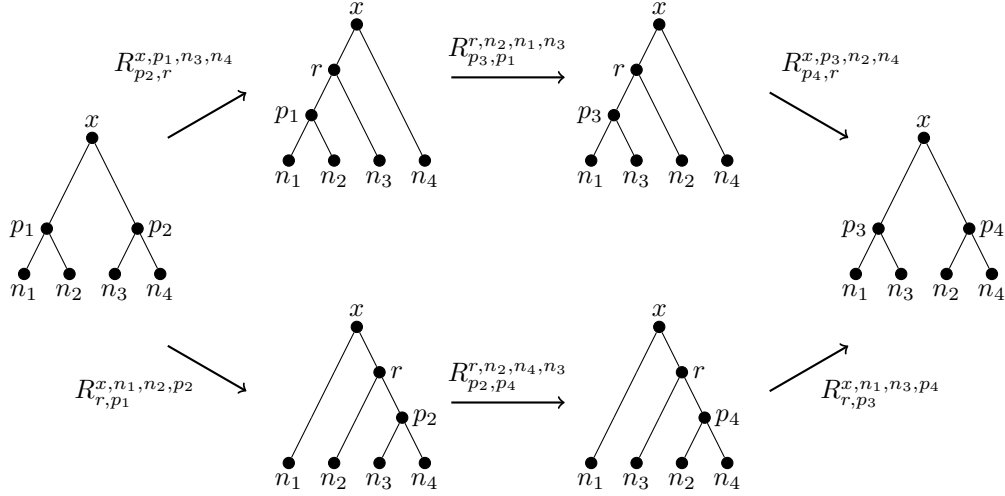
The second identity is



The corresponding identity for q -Bessel functions is obtained by substituting

$$\begin{aligned} r_1 - n_1 &= P, & p_1 - p_2 &= Q, & n_4 - r_2 &= R, & \nu &= x - n_1, \\ \mu_1 &= n_2 - p_1, & \mu_2 &= n_1 - p_2 + n_3 - n_4, & \mu &= p - n_4. \end{aligned}$$

The third identity is



□

Remark 3.7.

- (i) The q -Hankel transform of a function $f \in L^2(q^{\mathbb{Z}}; q^x)$ is defined by

$$(H_\nu f)(n) = \sum_{x \in \mathbb{Z}} f(q^x) J_\nu(q^{x+n}; q) q^x, \quad n \in \mathbb{Z}.$$

Identity (i) of Theorem 3.6 shows that the q -Hankel transform maps an orthogonal basis of q -Bessel functions to another orthogonal basis of q -Bessel functions, which implies a factorization of the q -Hankel transform: $H_{r_{123}} = H_{r_{312}} H_{r_{132}}$.

- (ii) Identity (ii), the product formula for q -Bessel functions, has appeared before in the literature; representation theoretic proofs are given by Koelink in [14, Cor.6.5], and Kalnins, Miller and Mukherjee in [11, (3.20)]. A direct analytic proof is given by Koelink and Swarttouw in [15].
- (iii) It is well known that the hexagon identity for classical $6j$ -symbols can be interpreted as a quantum Yang-Baxter equation. Here we obtain an infinite dimensional solution: for $u, v \in \mathbb{Z}$, define a unitary operator $\mathcal{R}(u, v) : \ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z})$ by

$$\mathcal{R}(u, v)(e_{x-a} \otimes e_{b-x}) = \sum_{y \in \mathbb{Z}} R_{x, y}^{u, a, v, b} e_{b-y} \otimes e_{y-a}, \quad a, b, x \in \mathbb{Z},$$

where $\{e_x \mid x \in \mathbb{Z}\}$ is the standard orthonormal basis for $\ell^2(\mathbb{Z})$. Then the hexagon identity says that the operator \mathcal{R} satisfies

$$\mathcal{R}_{12}(u, w) \mathcal{R}_{13}(v, w) \mathcal{R}_{23}(u, v) = \mathcal{R}_{23}(u, v) \mathcal{R}_{13}(v, w) \mathcal{R}_{12}(u, w)$$

as an operator identity on $\ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z})$.

4. 3nj-SYMBOLS AND MULTIVARIATE q -BESSEL FUNCTIONS

We consider certain 3nj-symbols and show that these can be considered as multivariate q -Bessel functions, which are limits of the multivariate Askey-Wilson polynomials introduced by Gasper and Rahman in [4]. In this section we use the following notation. For $v = (v_1, v_2, \dots, v_{d-1}, v_d)$ we define $|v| = \sum_{j=1}^d v_j$ and $\hat{v} = (v_d, v_{d-1}, \dots, v_2, v_1)$. For some function $f : \mathbb{Z}^d \rightarrow \mathbb{C}$ we set

$$\sum_x f(x) = \sum_{x_d \in \mathbb{Z}} \cdots \sum_{x_1 \in \mathbb{Z}} f(x_1, \dots, x_d),$$

provided the sum converges.

4.1. Multivariate q -Bessel functions. Let $d \in \mathbb{N}_{\geq 1}$. For $\nu = (\nu_0, \dots, \nu_{d+1}) \in \mathbb{Z}^{d+2}$, we define q -Bessel functions in the variables $x = (x_1, \dots, x_d), \lambda = (\lambda_1, \dots, \lambda_d) \in \mathbb{Z}^d$ by

$$(4.1) \quad J_\nu(x, \lambda) = \prod_{j=1}^d J_{\nu_j - x_{j+1} - \lambda_{j-1}}(q^{x_j - x_{j+1} + \lambda_j - \lambda_{j-1}}; q),$$

where $\lambda_0 = \nu_0$ and $x_{d+1} = \nu_{d+1}$. Occasionally we will use the notation $J_\nu(x, \lambda; q)$ to stress the dependence on q .

Theorem 4.1. *The multivariate q -Bessel functions have the following properties:*

(i) *Orthogonality relations:*

$$\sum_x J_\nu(x, \lambda) J_\nu(x, \lambda') q^{x_1} = \delta_{\lambda, \lambda'} q^{\nu_{d+1} + \nu_0 - \lambda_d}, \quad \lambda, \lambda' \in \mathbb{Z}^d.$$

(ii) *Self-duality:* $J_\nu(x, \lambda) = J_{\hat{\nu}}(\hat{\lambda}, \hat{x})$.

Proof. The self-duality property follows directly from (4.1). The orthogonality relations follow by induction using the q -Hankel orthogonality relations from Theorem 3.5, which can be written as

$$(4.2) \quad \sum_{x_j \in \mathbb{Z}} J_{\nu_j - x_{j+1} - \lambda_{j-1}}(q^{x_j - x_{j+1} + \lambda_j - \lambda_{j-1}}; q) J_{\nu_j - x_{j+1} - \lambda_{j-1}}(q^{x_j - x_{j+1} + \lambda'_j - \lambda_{j-1}}; q) q^{x_j} \\ = \delta_{\lambda_j, \lambda'_j} q^{x_{j+1} - \lambda_j + \lambda_{j-1}}.$$

Define for $k = 1, \dots, d+1$,

$$J_\nu^{(k)}(x, \lambda) = \prod_{j=k}^d J_{\nu_j - x_{j+1} - \lambda_{j-1}}(q^{\frac{1}{2}(x_j - x_{j+1} + \lambda_j - \lambda_{j-1})}; q),$$

the empty product being equal to 1. Note that $J_\nu^{(1)} = J_\nu$ and

$$(4.3) \quad J_{\nu_k - x_{k+1} - \lambda_{k-1}}(q^{x_k - x_{k+1} + \lambda_k - \lambda_{k-1}}; q) J_\nu^{(k+1)}(x, \lambda) = J_\nu^{(k)}(x, \lambda).$$

We will show that

$$(4.4) \quad \sum_{x_k \in \mathbb{Z}} \cdots \sum_{x_1 \in \mathbb{Z}} J_\nu(x, \lambda) J_\nu(x, \lambda') q^{x_1} = \delta_{\lambda_1, \lambda'_1} \cdots \delta_{\lambda_k, \lambda'_k} q^{x_{k+1} - \lambda_k + \lambda_0} J_\nu^{(k+1)}(x, \lambda) J_\nu^{(k+1)}(x, \lambda').$$

For $k = 1$ (4.4) follows directly from (4.2). Now assume that (4.4) holds for a certain k , then by (4.2) and (4.3),

$$\begin{aligned} \sum_{x_{k+1} \in \mathbb{Z}} \cdots \sum_{x_1 \in \mathbb{Z}} J_\nu(x, \lambda) J_\nu(x, \lambda') q^{x_1} \\ = \delta_{\lambda_1, \lambda'_1} \cdots \delta_{\lambda_k, \lambda'_k} \sum_{x_{k+1} \in \mathbb{Z}} J_\nu^{(k+1)}(x, \lambda) J_\nu^{(k+1)}(x, \lambda') q^{x_{k+1} - \lambda_k + \lambda_0} \\ = \delta_{\lambda_1, \lambda'_1} \cdots \delta_{\lambda_{k+1}, \lambda'_{k+1}} J_\nu^{(k+2)}(x, \lambda) J_\nu^{(k+2)}(x, \lambda') q^{x_{k+2} - \lambda_{k+1} + \lambda_0}, \end{aligned}$$

which proves the orthogonality relations. \square

Next we show that the multivariate q -Bessel functions can be considered as limit cases of multivariate Askey-Wilson polynomials. The 1-variable Askey-Wilson polynomials are defined by

$$p_n(x; a, b, c, d | q) = \frac{(ab, ac, ad; q)_n}{a^n} {}_4\phi_3 \left(\begin{matrix} q^{-n}, abcdq^{n-1}, ax, a/x \\ ab, ac, ad \end{matrix}; q, q \right),$$

which are polynomials in $x + x^{-1}$ of degree n , and they are symmetric in the parameters a, b, c, d . Using notation as in [8] the multivariable Askey-Wilson polynomials are defined as follows. Let $n = (n_1, \dots, n_d) \in \mathbb{N}^d$ and $x = (x_1, \dots, x_d) \in (\mathbb{C}^\times)^d$, then the d -variable Askey-Wilson polynomials are defined by

$$(4.5) \quad P_d(n; x; \alpha | q) = \prod_{j=1}^d p_{n_j} \left(x_j; \alpha_j q^{N_{j-1}}, \frac{\alpha_j}{\alpha_0^2} q^{N_{j-1}}, \frac{\alpha_{j+1}}{\alpha_j} x_{j+1}, \frac{\alpha_{j+1}}{\alpha_j} x_{j+1}^{-1} | q \right),$$

where $N_j = \sum_{k=1}^j n_k$, $N_0 = 0$, $\alpha = (\alpha_0, \dots, \alpha_{d+2}) \in \mathbb{C}^{d+3}$, $x_{d+1} = \alpha_{d+2}$. These are polynomials in the variables $x_1 + x_1^{-1}, \dots, x_d + x_d^{-1}$ of degree $|n| = N_d$.

Proposition 4.2. *Let $\lambda = (\lambda_1, \dots, \lambda_d) \in \mathbb{Z}^d$, $\nu = (\nu_0, \dots, \nu_{d+1}) \in \mathbb{Z}^{d+2}$ and define*

$$\begin{aligned} \alpha(m) &= \left(q^{-m}, q^{\frac{1}{2}\nu_0}, q^{\frac{1}{2}\nu_1-m}, \dots, q^{\frac{1}{2}\nu_j-jm}, \dots, q^{\frac{1}{2}\nu_d-dm}, q^{\nu_{d+1}+m} \right) \in \mathbb{C}^{d+3}, \\ x(m) &= \left(q^{-\frac{1}{2}\nu_0-x_1+m}, q^{\frac{1}{2}\nu_1-\nu_0-x_2+m}, \dots, q^{\frac{1}{2}\nu_{d-1}-\nu_0-x_d+m} \right) \in \mathbb{C}^d, \\ \lambda + m &= (\lambda_1 + m, \dots, \lambda_d + m) \in \mathbb{N}^d, \end{aligned}$$

$$C_m(x; \lambda; \alpha) = \prod_{j=1}^d q^{(\frac{1}{2}\nu_{j-1}-\nu_j+\nu_0+x_{j+1}-m)(\lambda_j+m)} (q^{\nu_j-\nu_{j-1}-2m}; q)_{\lambda_j+m},$$

then

$$\lim_{m \rightarrow \infty} \frac{P_d(\lambda + m; x(m); \alpha(m) | q)}{C_m(x; \lambda; \alpha)} = (q; q)_\infty^d \left(\prod_{j=1}^d q^{\frac{1}{2}(x_{j+1}-x_j+\Lambda_{j+1}-\Lambda_j)(\nu_j-x_{j+1}-\Lambda_{j-1})} \right) J_\nu(x, \Lambda),$$

where $\Lambda = (\Lambda_1, \dots, \Lambda_d)$ with $\Lambda_j = \nu_0 - \sum_{k=1}^j \lambda_k$ and $\Lambda_0 = \nu_0$.

Proof. First we substitute

$$\begin{aligned} \alpha_0 &\mapsto q^{-m}, & n_j &\mapsto \lambda_j + m, & j &= 1, \dots, d \\ x_j &\mapsto x_j q^m, & \alpha_j &\mapsto \alpha_j q^{m(j-1)}, & j &= 1, \dots, d+1, \end{aligned}$$

in (4.5) (recall, $x_{d+1} = \alpha_{d+2}$). The ${}_4\varphi_3$ -part of the j -th factor p_{n_j} is

$${}_4\varphi_3 \left(\begin{matrix} q^{-\lambda_j-m}, \alpha_{j+1}^2 q^{2(\sum_{k=1}^{j-1} \lambda_k) + \lambda_j - 1 + m}, \frac{\alpha_{j+1}}{\alpha_j} x_{j+1} x_j q^m, \frac{\alpha_{j+1} x_{j+1}}{\alpha_j x_j} q^{-m} \\ \frac{\alpha_{j+1}^2}{\alpha_j^2} q^{-2m}, \alpha_{j+1} x_{j+1} q^{2m + \sum_{k=1}^{j-1} \lambda_k}, \alpha_{j+1} x_{j+1} q^{\sum_{k=1}^{j-1} \lambda_k} \end{matrix} ; q, q \right),$$

where the empty sum equals 0. Letting $m \rightarrow \infty$ this function tends to

$${}_1\varphi_1 \left(\begin{matrix} 0 \\ \alpha_{j+1} x_{j+1} q^{\sum_{k=1}^{j-1} \lambda_k} \end{matrix} ; q, \frac{\alpha_j x_{j+1}}{\alpha_{j+1} x_j} q^{1-\lambda_j} \right).$$

Finally we substitute

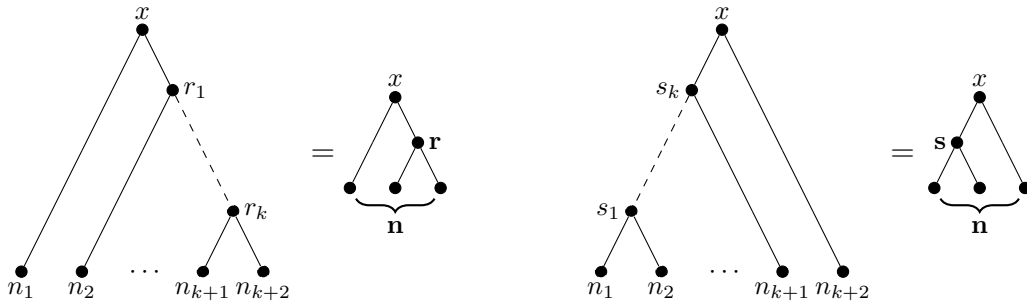
$$\alpha_j \mapsto q^{\frac{1}{2}\nu_{j-1}}, \quad x_j \mapsto q^{\frac{1}{2}\nu_{j-1} - \nu_0 - x_j},$$

and set $\nu_0 - \sum_{k=1}^j \lambda_k = \Lambda_j$ for $j = 0, \dots, d$, then we have

$${}_1\varphi_1 \left(\begin{matrix} 0 \\ q^{\nu_j - x_{j+1} - \Lambda_{j-1}} \end{matrix} ; q, q^{x_{j+1} - x_j + \Lambda_j - \Lambda_{j-1}} \right),$$

which we recognize as the ${}_1\varphi_1$ -part of the j -th factor of the multivariate q -Bessel function $J_\nu(x, \Lambda)$, see (4.1). \square

4.2. 3nj-symbols. Let $k \in \mathbb{N}_{\geq 1}$, and let $\mathbf{r}, \mathbf{s} \in \mathbb{Z}^k$, $\mathbf{n} \in \mathbb{Z}^{k+2}$. We define the 3nj-symbols $R_{\mathbf{r}, \mathbf{s}}^{x; \mathbf{n}}$ to be the coupling coefficients between two specific binary trees corresponding to $(k+2)$ -fold tensor product representations. We will use the following notation:



Note that a node with a bold symbol represents several nodes, and that the label \mathbf{r} (respectively \mathbf{s}) on the right (left) of a node means that all branches ‘hang’ on the right (left) edge. The $3nj$ -symbols $R_{\mathbf{r},\mathbf{s}}^{x,\mathbf{n}}$ are defined by

$$\text{Diagram with } x \text{ at top, } \mathbf{n} \text{ at bottom, } \mathbf{r} \text{ on right edge} = \sum_{\mathbf{s}} R_{\mathbf{r},\mathbf{s}}^{x,\mathbf{n}} \text{Diagram with } x \text{ at top, } \mathbf{n} \text{ at bottom, } \mathbf{s} \text{ on left edge}$$

and we will denote the corresponding transition again by an arrow. Note that for $k = 1$ we have $R_{r,s}^{x,\mathbf{n}} = R_{r,s}^{x,n_1,n_2,n_3}$.

Proposition 4.3. *The coefficients $R_{\mathbf{r},\mathbf{s}}^{x,\mathbf{n}}$ have the following properties:*

(i) *Orthogonality relations:* $\sum_{\mathbf{r}} R_{\mathbf{r},\mathbf{s}}^{x,\mathbf{n}} R_{\mathbf{r},\mathbf{s}'}^{x,\mathbf{n}} = \delta_{\mathbf{s},\mathbf{s}'}$

(ii) *Duality:* $R_{\mathbf{r},\mathbf{s}}^{x,\mathbf{n}} = R_{\hat{\mathbf{s}},\hat{\mathbf{r}}}^{x,\hat{\mathbf{n}}}$.

Proof. The coefficients R are the matrix coefficients of a unitary operator, which implies the orthogonality relations. The duality property is a consequence of the identity

$$\text{Diagram with } x \text{ at top, } \mathbf{n} \text{ at bottom, } \mathbf{r} \text{ on right edge} = \text{Diagram with } x \text{ at top, } \hat{\mathbf{n}} \text{ at bottom, } \hat{\mathbf{r}} \text{ on left edge}$$

which follows from repeated application of (3.2). □

Theorem 4.4. *For $i = 1, 2$ let $k_i \in \mathbb{N}_{\geq 1}$, $\mathbf{n}_i \in \mathbb{Z}^{k_i+1}$ and $\mathbf{r}_i, \mathbf{s}_i \in \mathbb{Z}^{k_i}$. Let $k = k_1 + k_2$, $\mathbf{n} = (\mathbf{n}_1, \mathbf{n}_2)$, $\mathbf{r} = (\mathbf{r}_1, \mathbf{r}_2)$, $\mathbf{s} = (\mathbf{s}_1, \mathbf{s}_2)$, then*

$$R_{\mathbf{r},\mathbf{s}}^{x,\mathbf{n}} = R_{\mathbf{r}_1,\mathbf{s}_1}^{x,(\mathbf{n}_1,r_{k_1+1})} R_{\mathbf{r}_2,\mathbf{s}_2}^{x,(s_{k_1},\mathbf{n}_2)}.$$

As a consequence,

$$R_{\mathbf{r},\mathbf{s}}^{x,\mathbf{n}} = \prod_{j=1}^k R_{r_j,s_j}^{x,s_{j-1},n_{j+1},r_{j+1}},$$

where $s_0 = n_1$ and $r_{k+1} = n_{k+2}$.

Proof. The first identity follows from

$$\text{Diagram 1} \xrightarrow{R_{\mathbf{r}_1,\mathbf{s}_1}^{x,(\mathbf{n}_1,r_{k_1+1})}} \text{Diagram 2} \xrightarrow{R_{\mathbf{r}_2,\mathbf{s}_2}^{x,(s_{k_1},\mathbf{n}_2)}} \text{Diagram 3}$$

The second identity follows from repeated application of the first identity. □

From (3.3) it follows that $R_{\mathbf{r},\mathbf{s}}^{x,\mathbf{n}}$ is essentially a multivariate q -Bessel function as defined by (4.1).

Corollary 4.5. *Let $\nu(x, \mathbf{n}) = (n_1, x + n_2, \dots, x + n_{k+1}, n_{k+2})$, then*

$$R_{\mathbf{r},\mathbf{s}}^{x,\mathbf{n}} = (-q)^{r_1+s_k-n_1-n_{k+2}} J_{\nu(x,\mathbf{n})}(\mathbf{r}, \mathbf{s}; q^2).$$

Note that this corollary and Proposition 4.3 together give a representation theoretic proof of Theorem 4.1

Our next goal is to prove a summation identity for the multivariate q -Bessel functions. Let us first mention that by interpreting a binary tree as a product of Clebsch-Gordan coefficients, the $3nj$ -symbols $R_{\mathbf{r},\mathbf{s}}^{x,\mathbf{n}}$ satisfy, by definition, the formula

$$(4.6) \quad C_{x,\mathbf{r},\mathbf{n}} = \sum_{\mathbf{s}} R_{\mathbf{r},\mathbf{s}}^{x,\mathbf{n}} C_{x,\hat{\mathbf{s}},\hat{\mathbf{n}}}, \quad C_{x,\mathbf{r},\mathbf{n}} = \prod_{j=1}^{k+1} C_{r_{j-1},n_j,r_j},$$

where $r_0 = x, r_{k+1} = n_{k+2}, s_0 = n_1, s_{k+2} = x$. The functions $C_{x,\mathbf{r},\mathbf{n}}$ can be considered as multivariate Wall polynomials, which are q -analogs of Laguerre polynomials. In this light (4.6) is a multivariate q -analog of an identity proved by Erdélyi [2] which states that the Hankel transform maps a product of two Laguerre polynomials to a product of two Laguerre polynomials.

For the $3nj$ -symbols $R_{\mathbf{r},\mathbf{s}}^{x,\mathbf{n}}$ there exists a multivariate analog of the Biedenharn-Elliott identity. In terms of q -Bessel functions this gives an expansion formula for k -variable q -Bessel functions in terms of $(k-1)$ -variable q -Bessel functions. The identity requires also another $3nj$ -symbol. For $\mathbf{r}, \mathbf{s} \in \mathbb{Z}^k$, $\mathbf{n} \in \mathbb{Z}^{k+2}$, $x \in \mathbb{Z}$, let $S_{\mathbf{r},\mathbf{s}}^{x,\mathbf{n}}$ be the coupling coefficient defined by

$$(4.7) \quad \begin{array}{c} x \\ \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \underbrace{\hspace{1.5cm}}_{\mathbf{n}} \end{array} = \sum_{\hat{\mathbf{s}}} S_{\mathbf{r},\mathbf{s}}^{x,\mathbf{n}} \begin{array}{c} x \\ \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \underbrace{\hspace{1.5cm}}_{\mathbf{n}} \end{array}$$

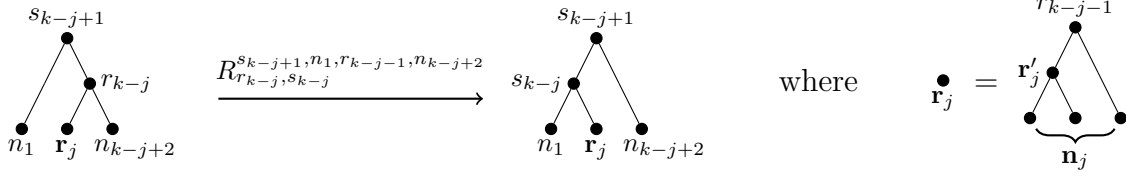
Note that $\sum_{\hat{\mathbf{s}}} = \sum_{s_1} \cdots \sum_{s_k}$. This $3nj$ -symbol can of course also be considered as a multivariate q -Bessel function (see the following result), but it lacks the self-duality property. Let us first express S in terms of the $6j$ -symbols.

Lemma 4.6. $S_{\mathbf{r},\mathbf{s}}^{x,\mathbf{n}}$ is given by

$$S_{\mathbf{r},\mathbf{s}}^{x,\mathbf{n}} = \prod_{j=1}^k R_{r_j, s_j}^{s_{j+1}, n_1, r_{j-1}, n_{j+2}},$$

with $s_{k+1} = x$ and $r_0 = n_2$.

Proof. We use the transition



and where $\mathbf{r}'_j = (r_1, \dots, r_{k-j-2})$ and $\mathbf{n}_j = (n_2, \dots, n_{k-j+1})$. We set $s_{k+1} = x$ and $r_0 = n_2$, then applying this transition successively on subtrees for $j = 0, \dots, k-1$ gives

$$S_{\mathbf{r}, \mathbf{s}}^{x, \mathbf{n}} = \prod_{j=0}^{k-1} R_{r_{k-j}, s_{k-j}}^{s_{k-j+1}, n_1, r_{k-j-1}, n_{k-j+2}}.$$

Changing the index gives the stated expression for the coupling coefficient S . \square

The following identity is the multivariate analog of the Biedenharn-Elliott identity from Theorem 3.6, i.e., the $k = 2$ case gives back the Theorem 3.6(ii).

Theorem 4.7. For $k \in \mathbb{N}_{\geq 2}$ let $\mathbf{r}, \mathbf{s} \in \mathbb{Z}^k$ and $\mathbf{n} \in \mathbb{Z}^{k+2}$, then

$$R_{\mathbf{r}, \mathbf{s}}^{x, \mathbf{n}} = \sum_{\mathbf{t} \in \mathbb{Z}^{k-1}} S_{(\mathbf{t}, r_1), \mathbf{s}}^{x, \mathbf{n}} R_{\mathbf{r}', \mathbf{t}}^{r_1, \mathbf{n}'},$$

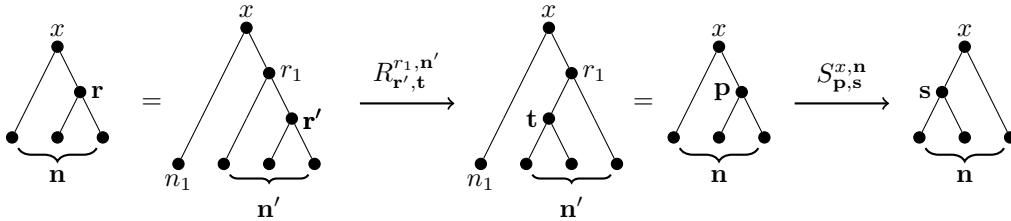
where \mathbf{v}' is obtained from \mathbf{v} by leaving out the first component. In terms of multivariate q -Bessel functions,

$$J_{\nu(x, \mathbf{n})}(\mathbf{r}, \mathbf{s}) = \sum_{\mathbf{t} \in \mathbb{Z}^{k-1}} A_{\mathbf{t}, \mathbf{s}}^{r_1} J_{\nu(r_1, \mathbf{n}')}(\mathbf{r}', \mathbf{t})$$

with

$$A_{\mathbf{t}, \mathbf{s}}^{r_1} = (-q^{\frac{1}{2}})^{|\mathbf{t}| + |\mathbf{s}| - |\mathbf{n}| - (k-2)n_1 - s_k + r_2} \prod_{j=1}^k J_{s_{j+1} - n_1 + t_{j-1} + n_{j+2}}(q^{s_j + t_j - n_1 - n_{j+2}}; q), \quad t_k = r_1.$$

Proof. This follows from the transition



where $\mathbf{p} = (\mathbf{t}, r_1)$, and the definition of the coupling coefficients R . \square

Remark 4.8. It seems that there are no analogs for the $3nj$ -symbols R of identities (i) and (iii) of Theorem 3.6, but there does exist an analog of Theorem 3.6(i) involving only the $3nj$ -symbols S which may be of interest. This is obtained as follows.

Let $\mathbf{n} \in \mathbb{Z}^{k+2}$. For $j \in \{1, 2, \dots, k+1\}$ we define $\mathbf{n}_j = (n_{k+3-j}, \dots, n_{k+2}, n_1, \dots, n_{k+2-j})$. Furthermore, given a vector \mathbf{v} , we denote (as in Theorem 4.7) by \mathbf{v}' the vector \mathbf{v} without the first component, and we set $\mathbf{n}'_j = (\mathbf{n}_j)'$. Consider the transition

Iterating this transition $k+1$ times shows that the coupling coefficient in the transition

is given by

$$T_{\mathbf{r}, \mathbf{s}}^{x, \mathbf{n}} = \sum_{\mathbf{s}_k} \cdots \sum_{\mathbf{s}_1} \left(S_{\mathbf{s}_0, \mathbf{s}_1}^{x, \mathbf{n}_1} \cdots S_{\mathbf{s}_k, \mathbf{s}_{k+1}}^{x, \mathbf{n}_{k+1}} \right), \quad \mathbf{s}_0 = \mathbf{r}, \quad \mathbf{s}_{k+1} = \mathbf{s}.$$

On the other hand, by the definition of the coupling coefficient S we have

so that

$$S_{\mathbf{s}, \mathbf{r}}^{x, \mathbf{n}} = \sum_{\mathbf{s}_k} \cdots \sum_{\mathbf{s}_1} \left(S_{\mathbf{s}_0, \mathbf{s}_1}^{x, \mathbf{n}_1} \cdots S_{\mathbf{s}_k, \mathbf{s}_{k+1}}^{x, \mathbf{n}_{k+1}} \right), \quad \mathbf{s}_0 = \mathbf{r}, \quad \mathbf{s}_{k+1} = \mathbf{s}.$$

For $k=1$ this gives back Theorem 3.6(i).

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